Brownian motion in a magnetic field

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We derive explicit forms of Markovian transition probability densities for the velocity-space, phase-space, and the Smoluchowski configuration-space Brownian motion of a charged particle in a constant magnetic field. By invoking a hydrodynamical formalism for those stochastic processes, we quantify a continual (net on the local average) heat transfer from the thermostat to diffusing particles.

I. INTRODUCTION

We address an old-fashioned problem of the Brownian motion of a charged particle in a constant magnetic field. This issue has originated from studies of the diffusion of plasma across a magnetic field [1,2] and nowadays, together with a free Brownian motion example, stands for a textbook illustration of how transport and autocorrelation functions should be computed in generic situations governed by the Langevin equation (to a suitable degree of approximation of a kinetic theory, when collisions are stochastically modeled in terms of a random force), cf. [3] but also [4,5].

From a purely pragmatic point of view, this white-noise strategy is quite satisfactory. After (formally) evaluating velocity autocorrelation functions, formulas for running and asymptotic diffusion coefficients easily follow. To that end an explicit form of the probability density or transition probability density of the involved stochastic diffusion processes (in velocity space, space or configuration space) is not necessary, cf. [1,3].

To our knowledge, except for [2] (mentioned in [5] as a footnote reference for the purpose of evaluation of the mean square velocity and its mean square displacement at equilibrium), for a Brownian particle in a constant magnetic field, no attempt was made in the literature to give a complete characterization of the stochastic process itself, nor pass to the associated macrosopic (hydrodynamical formalism) balance equations (cf. [6–11] for a number of reasons why to do that).

Surprisingly enough, in Ref. [2], the Brownian motion in a magnetic field is described in terms of *operator-valued* (matrix-valued functions) probability distributions that additionally involve fractional powers of matrices. In consequence, there is no clean path towards a (necessary) relationship with the associated Kramers-Smoluchowski equations (cf. Chap. 6.1 in Ref. [4]), nor ways to stay in conformity with the standard wisdom about probabilistic procedures valid in case of the free Brownian motion (Ornstein-Uhlenbeck process), cf. [12,13,9].

Therefore, we decided to address an issue of the Brownian motion in a magnetic field anew, to unravel its features of a fully fledged stochastic diffusion process. In particular, we derive transition probability densities governing the velocity-space, phase-space, and the configuration-space processes. Hydrodynamical balance equations and their behavior in the Smoluchowski regime are discussed as well.

II. VELOCITY-SPACE DIFFUSION PROCESS

The standard analysis of the Brownian motion of a free particle employs the Langevin equation $d\vec{u}/dt = -\beta\vec{u} + \vec{A}(t)$ where $\vec{u}$ denotes the velocity of the particle and the influence of the surrounding medium on the motion (random acceleration) of the particle is modeled by means of two independent contributions. A systematic part $-\beta\vec{u}$ represents a dynamical friction. The remaining fluctuating part $\vec{A}(t)$ is supposed to display a statistic of the familiar white noise: (i) $\vec{A}(t)$ is independent of $\vec{u}$, (ii) $\langle A_i(s)\rangle = 0$ and $\langle A_i(s)A_j(s')\rangle = 2q\,\delta_{ij}\delta(s-s')$ for $i,j = 1,2,3$, where $q = (k_BT/m)\beta$ is a physical parameter. The well-known Ornstein-Uhlenbeck stochastic process comes out on that conceptual basis [12,13,9].

The linear friction model can be adopted to the case of diffusion of charged particles in the presence of a constant magnetic field that acts upon particles via the Lorentz force. The Langevin equation for that motion reads

$$\frac{d\vec{u}}{dt} = -\beta\vec{u} + \frac{q_e}{mc}\vec{u}\times\vec{B} + \vec{A}(t), \qquad (1)$$

where q_e denotes an electric charge of the particle of mass m.

Let us assume for simplicity that the constant magnetic field $\vec{B}$ is directed along the z axis of a Cartesian reference frame: $\vec{B} = (0,0,B)$ and $B = \text{const}$. In this case Eq. (1) takes the form

$$\frac{d\vec{u}}{dt} = -\Lambda\vec{u} + \vec{A}(t), \qquad (2)$$

where

$$\Lambda = \begin{pmatrix} \beta & -\omega_c & 0 \\ \omega_c & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}, \tag{3}$$

and $\omega_c = q_e B/mc$ denotes the Larmor frequency. Assuming the Langevin equation to be (at least formally) solvable, we can infer a probability density $P(\vec{u},t|\vec{u}_0), t>0$ conditioned by initial velocity data choice $\vec{u}=\vec{u}_0$ at $t=0$. Physical circumstances of the problem enforce a demand: (i) $P(\vec{u},t|\vec{u}_0) \rightarrow \delta^3(\vec{u}-\vec{u}_0)$ as $t\rightarrow 0$ and (ii) $P(\vec{u},t|\vec{u}_0) \rightarrow (m/2\pi k_B T)^{3/2} \exp(-m|\vec{u}_0|^2/2k_B T)$ as $t\rightarrow\infty$.

A formal solution of Eq. (2) reads

$$\vec{u}(t) - e^{-\Lambda t}\vec{u}_0 = \int_0^t e^{-\Lambda(t-s)}\vec{A}(s)ds. \tag{4}$$

By taking into account that

$$e^{-\Lambda t} = e^{-\beta t} \begin{pmatrix} \cos\omega_c t & \sin\omega_c t & 0 \\ -\sin\omega_c t & \cos\omega_c t & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{-\beta t}U(t), \tag{5}$$

we can rewrite Eq. (4) as follows:

$$\vec{u}(t) - e^{-\beta t}U(t)\vec{u}_0 = \int_0^t e^{-\beta(t-s)}U(t-s)\vec{A}(s)ds. \tag{6}$$

Statistical properties of $\vec{u}(t) - e^{-\Lambda t}\vec{u}_0$ are identical with those of $\int_0^t e^{-\Lambda(t-s)}\vec{A}(s)ds$. In consequence, the problem of deducing a probability density $P(\vec{u},t|\vec{u}_0)$ is equivalent to deriving the probability distribution of the random vector

$$\vec{S} = \int_0^t \psi(s)\vec{A}(s)ds, \tag{7}$$

where $\psi(s) = e^{-\Lambda(t-s)} = e^{-\beta(t-s)}U(t-s)$.

The white-noise term $\vec{A}(s)$, in view of the integration with respect to time, is amenable to a more rigorous analysis that invokes the Wiener process increments and their statistics [14]. Let us divide the time integration interval into a large number of small subintervals Δt. We adjust them suitably to assure that effectively $\psi(s)$ is constant on each subinterval $(j\Delta t, \{j+1\}\Delta t)$ and equal $\psi(j\Delta t)$. As a result we obtain the expression

$$\vec{S} = \sum_{j=0}^{N-1} \psi(j\Delta t) \int_{j\Delta t}^{(j+1)\Delta t} \vec{A}(s)ds. \tag{8}$$

Here $\vec{B}(\Delta t) = \int_{j\Delta t}^{(j+1)\Delta t}\vec{A}(s)ds$ stands for the abovementioned Wiener process increment. Physically, $\vec{B}(\Delta t)$ represents the *net* acceleration that a Brownian particle may suffer (in fact accumulates) during an interval of time Δt.

Equation (8) becomes

$$\vec{S} = \sum_{j=0}^{N-1} \psi(j\Delta t)\vec{B}(\Delta t) = \sum_{j=0}^{N-1} \vec{s}_j, \tag{9}$$

where we introduce $\vec{s}_j = \psi(j\Delta t)\vec{B}(\Delta t) = \psi_j\vec{B}(\Delta t)$.

The Wiener process argument [12,13,6] allows us to infer the probability distribution of $\vec{s}_j$. It is enough to employ the fact that the distribution of $\vec{B}(\Delta t)$ is Gaussian with mean zero and variance $q = (k_B T/m)\beta$. Then

$$w[\vec{B}(\Delta t)] = \left(\frac{1}{4\pi q\Delta t}\right)^{3/2} \exp\left(-\frac{|\vec{B}(\Delta t)|^2}{4q\Delta t}\right) \tag{10}$$

and in view of $\vec{s}_j = \psi_j\vec{B}(\Delta t)$ by performing the change of variables in Eq. (10), we get

$$\tilde{w}[\vec{s}_j] = \det[\psi_j^{-1}]w[\psi_j^{-1}\vec{s}_j] = \frac{1}{\det\psi_j}w[\psi_j^{-1}\vec{s}_j]. \tag{11}$$

Since $\det\psi(s) = e^{-3\beta(t-s)}$ and $\psi^{-1}(s) = U[-(t-s)]e^{\beta(t-s)}$ we obtain

$$\tilde{w}[\vec{s}_j] = \left(\frac{1}{4\pi q\Delta t}\right)^{3/2} \frac{1}{e^{-3\beta(t-j\Delta t)}}$$

$$\times \exp\left(-\frac{|e^{\beta(t-j\Delta t)}U[-(t-j\Delta t)]\vec{s}_j|^2}{4q\Delta t}\right) \tag{12}$$

and finally

$$\tilde{w}[\vec{s}_j] = \left(\frac{1}{4\pi q\Delta t}\frac{1}{e^{-2\beta(t-j\Delta t)}}\right)^{3/2} \exp\left(-\frac{|\vec{s}_j|^2}{4q\Delta t e^{-2\beta(t-j\Delta t)}}\right). \tag{13}$$

Clearly, $\vec{s}_j$ are mutually independent random variables whose distribution is Gaussian with mean zero and variance $\sigma_j^2 = 2q\Delta t e^{-2\beta(t-j\Delta t)}$. Hence, the probability distribution of $\vec{S} = \Sigma_{j=0}^{N-1}\vec{s}_j$ is again Gaussian with mean zero. Its variance equals the sum of variances of $\vec{s}_j$ i.e., $\sigma^2 = \Sigma_j\sigma_j^2 = 2q\Sigma_j\Delta t e^{-2\beta(t-j\Delta t)}$.

Taking the limit $N\rightarrow\infty$ ($\Delta t\rightarrow 0$) we arrive at

$$\sigma^2 = 2q\int_0^t ds e^{-2\beta(t-s)} = \frac{k_B T}{m}(1-e^{-2\beta t}). \tag{14}$$

Because of $\vec{S} = \vec{u}(t) - e^{-\Lambda t}\vec{u}_0$ the transition probability density of the Brownian particle velocity, conditioned by the initial data $\vec{u}_0$ at $t_0 = 0$ reads

$$P(\vec{u},t|\vec{u}_0) = \left(\frac{1}{2\pi\frac{k_B T}{m}(1-e^{-2\beta t})}\right)^{3/2}$$

$$\times \exp\left(-\frac{|\vec{u}-e^{-\Lambda t}\vec{u}_0|^2}{2\frac{k_B T}{m}(1-e^{-2\beta t})}\right). \tag{15}$$

The process is Markovian and time homogeneous, hence the above formula can be trivially extended to encompass the case of arbitrary $t_0 \neq 0$: $P(\vec{u}, t | \vec{u}_0, t_0)$ arises by substituting everywhere $t - t_0$ instead of t.

Physical arguments [cf. demand (ii) preceding Eq. (4)] refer to an asymptotic probability distribution (invariant measure density) $P(u)$ of the random variable $\vec{u}$ in the Maxwell-Boltzmann form

$$P(\vec{u}) = \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m|\vec{u}|^2}{2k_B T} \right). \quad (16)$$

This time-independent probability density together with the

time-homogeneous transition density (15) uniquely determine a stationary Markovian stochastic process for which we can evaluate various mean values.

Expectation values of velocity components vanish: $\langle u_i(t) \rangle = \int_{-\infty}^{\infty} u_i P(\vec{u}) d\vec{u} = 0$ for $i = 1,2,3$. The matrix of the second moments (velocity autocorrelation functions) reads

$$\langle u_i(t) u_j(t_0) \rangle = \int_{-\infty}^{\infty} u_i u_j^0 P(\vec{u}, t; \vec{u}_0, t_0) d\vec{u} d\vec{u}_0, \quad (17)$$

where $i, j = 1,2,3$ and in view of $P(\vec{u}, t; \vec{u}_0, t_0) = P(\vec{u}, t | \vec{u}_0, t_0) P(\vec{u}_0)$ we arrive at the compact expression

$$\frac{k_B T}{m} e^{-\Lambda|t-t_0|} = \frac{k_B T}{m} e^{-\beta|t-t_0|} \begin{pmatrix} \cos\omega_c|t-t_0| & \sin\omega_c|t-t_0| & 0 \\ -\sin\omega_c|t-t_0| & \cos\omega_c|t-t_0| & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

In particular, the autocorrelation function (second moment) of the x component of velocity equals

$$\langle u_1(t) u_1(t_0) \rangle = \frac{k_B T}{m} e^{-\beta|t-t_0|} \cos\omega_c|t-t_0|, \quad (19)$$

in agreement with white-noise calculations of Refs. [1] and [3], cf. Chap. 11, formula (11.25). In particular, the so-called running diffusion coefficient arises via straightforward integration of the function $R_{11}(\tau) = \langle u_1(t) u_1(t_0) \rangle$ where $\tau = t - t_0 > 0$:

$$D_1(t) = \int_0^t \langle u_1(0) u_1(\tau) \rangle d\tau$$

$$= \frac{k_B T}{m} \frac{\beta + [\omega_c \sin(\omega_c t) - \beta \cos(\omega_c t)] \exp(-\beta t)}{\beta^2 + \omega_c^2}$$

$$(20)$$

with an obvious asymptotics [the same for $D_2(t)$]: $D_B = \lim_{t \to \infty} D_1(t) = (k_B T/m)[\beta/(\beta^2 + \omega_c^2)]$ and the large friction (ω_c fixed and bounded) version $D = k_B T/m\beta$.

III. SPATIAL PROCESS

The cylindrical symmetry of the problem allows us to consider separately processes running on the XY plane and along the Z axis (where the free Brownian motion takes place). We shall confine further attention to the two-dimensional XY plane problem. Henceforth, each vector will carry two components that correspond to the x and y coordinates, respectively. We will directly refer to the vector and matrix quantities introduced in the previous section, but while keeping the same notation, we shall simply disregard their z-coordinate contributions.

We define the spatial displacement $\vec{r}$ of the Brownian particle as follows:

$$\vec{r} - \vec{r}_0 = \int_0^t \vec{u}(\eta) dn, \quad (21)$$

where $\vec{u}(t)$ is given by Eq. (2) (except for disregarding the third coordinate).

Our aim is to derive the probability distribution of $\vec{r}$ at time t provided that the particle position and velocity were equal to $\vec{r}_0$ and $\vec{u}_0$, respectively, at time $t_0 = 0$. To that end we shall mimic procedures of the previous section. In view of

$$\vec{r} - \vec{r}_0 - \int_0^t e^{-\Lambda \eta} \vec{u}_0 = \int_0^t d\eta \int_0^\eta ds\, e^{-\Lambda(\eta-s)} \vec{A}(s), \quad (22)$$

we have

$$\vec{r} - \vec{r}_0 - \Lambda^{-1}(1 - e^{-\Lambda t}) \vec{u}_0 = \int_0^t \Lambda^{-1}(1 - e^{\Lambda(s-t)}) \vec{A}(s) ds, \quad (23)$$

where

$$\Lambda^{-1} = \frac{1}{\beta^2 + \omega_c^2} \begin{pmatrix} \beta & \omega_c \\ -\omega_c & \beta \end{pmatrix} \quad (24)$$

is the inverse of the matrix Λ [regarded as a rank two sub-matrix of that originally introduced in Eq. (3)]. Let us define two auxiliary matrices

$$\Omega \equiv \Lambda^{-1}(1 - e^{-\Lambda t}), \quad (25)$$

$$\phi(s) \equiv \Lambda^{-1}(1 - e^{\Lambda(s-t)}).$$

Because of

$$e^{-\Lambda t} = \exp\left\{-t\begin{pmatrix} \beta & -\omega_c \\ \omega_c & \beta \end{pmatrix}\right\} = e^{-\beta t}\begin{pmatrix} \cos\omega_c t & \sin\omega_c t \\ -\sin\omega_c t & \cos\omega_c t \end{pmatrix}$$

$$= e^{-\beta t} U(t) \tag{26}$$

we can represent matrices Ω, $\phi(s)$ in more detailed form. We have

$$\phi(s) = \Lambda^{-1}[1 - e^{-\beta(t-s)}U(t-s)] = \frac{1}{\beta^2 + \omega_c^2}\begin{pmatrix} \beta & \omega_c \\ -\omega_c & \beta \end{pmatrix}\begin{pmatrix} 1 - e^{\beta(s-t)}\cos\omega_c(s-t) & -e^{\beta(s-t)}\sin\omega_c(s-t) \\ e^{\beta(s-t)}\sin\omega_c(s-t) & 1 - e^{\beta(s-t)}\cos\omega_c(s-t) \end{pmatrix}. \tag{28}$$

Next steps imitate procedures of the previous section. Thus, we seek for the probability distribution of the random (planar) vector $\vec{R} = \int_0^t \phi(s)\vec{A}(s)ds$ where $\vec{R} = \vec{r} - \vec{r}_0 - \Omega\vec{u}_0$.

Dividing the time interval $(0,t)$ into small subintervals to assure that $\phi(s)$ can be regarded constant over the time span $(j\Delta t, \{j+1\}\Delta t)$ and equal $\phi(j\Delta t)$, we obtain

$$\vec{R} = \sum_{j=0}^{N-1} \phi(j\Delta t)\int_{j\Delta t}^{(j+1)\Delta t} \vec{A}(s)ds$$

$$= \sum_{j=0}^{N-1} \phi(j\Delta t)\vec{B}(\Delta t)$$

$$= \sum_{j=0}^{N-1} \vec{r}_j, \tag{29}$$

where $\vec{r}_j = \phi(j\Delta t)\vec{B}(\Delta t) = \phi_j\vec{B}(\Delta t)$.

By invoking the probability distribution (10) we perform an appropriate change of variables: $\vec{r}_j = \phi_j\vec{B}(\Delta t)$ to yield a probability distribution of $\vec{r}_j$

$$\widetilde{w}[\vec{r}_j] = \det[\phi_j^{-1}]w[\phi_j^{-1}\vec{r}_j] = \frac{1}{\det\phi_j}w[\phi_j^{-1}\vec{r}_j]. \tag{30}$$

$$\widetilde{A}_j = \begin{pmatrix} 1 - e^{\beta(j\Delta t - t)}\cos\omega_c(j\Delta t - t) & e^{\beta(j\Delta t - t)}\sin\omega_c(j\Delta t - t) \\ -e^{\beta(j\Delta t - t)}\sin\omega_c(j\Delta t - t) & 1 - e^{\beta(j\Delta t - t)}\cos\omega_c(j\Delta t - t) \end{pmatrix}\begin{pmatrix} \beta & -\omega_c \\ \omega_c & \beta \end{pmatrix} \tag{34}$$

and

$$\gamma_j = 1 + e^{2\beta(j\Delta t - t)} - 2e^{\beta(j\Delta t - t)}\cos\omega_c(j\Delta t - t). \tag{35}$$

There holds

$$\det\phi_j^{-1} = (\det\phi_j)^{-1} = (\beta^2 + \omega_c^2)\frac{1}{\gamma_j} \tag{36}$$

$$\Omega = \frac{1}{\beta^2 + \omega_c^2}\left\{\begin{pmatrix} \beta & \omega_c \\ -\omega_c & \beta \end{pmatrix} - e^{-\beta t}\begin{pmatrix} \beta & \omega_c \\ -\omega_c & \beta \end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix} \cos\omega_c t & \sin\omega_c t \\ -\sin\omega_c t & \cos\omega_c t \end{pmatrix}\right\} \tag{27}$$

and

Presently [not to be confused with previous steps (11)–(15)] we have

$$\det\phi(s) = \frac{1}{\beta^2 + \omega_c^2}\{1 + e^{2\beta(s-t)} - 2e^{\beta(s-t)}\cos\omega_c(s-t)\} \tag{31}$$

and

$$\phi^{-1}(s) = \frac{1}{1 + e^{2\beta(s-t)} - 2e^{\beta(s-t)}\cos\omega_c(s-t)}$$

$$\times[1 - e^{\beta(s-t)}U\{-(s-t)\}]\Lambda. \tag{32}$$

So, the inverse of the matrix ϕ_j has the form

$$\phi_j^{-1} = \frac{\widetilde{A}_j}{\gamma_j} \tag{33}$$

where

and as a consequence the probability distribution of $\vec{r}_j$ becomes

$$\widetilde{w}[\vec{r}_j] = \frac{1}{\frac{1}{\beta^2 + \omega_c^2}\gamma_j}\left(\frac{1}{4\pi q\Delta t}\right)\exp\left\{\frac{\left|\widetilde{A}_j\begin{pmatrix} r_j^x \\ r_j^y \end{pmatrix}\right|^2}{\gamma_j^2 4q\Delta t}\right\}. \tag{37}$$

In view of

$$\left|\widetilde{A}_j\begin{pmatrix} r_j^x \\ r_j^x \end{pmatrix}\right|^2 = (\beta^2 + \omega_c^2)\gamma_j[(r_j^x)^2 + (r_j^y)^2] \qquad (38)$$

that finally leads to

$$\widetilde{w}[\vec{r}_j] = \left(\frac{\beta^2 + \omega_c^2}{4\pi q \Delta t \gamma_j}\right) \exp\left\{-\frac{(\beta^2 + \omega_c^2)|\vec{r}_j|^2}{4q\Delta t \gamma_j}\right\}. \qquad (39)$$

Since this probability distribution is Gaussian with mean zero and variance $\sigma_j^2 = 2q\Delta t[1/(\beta^2 + \omega_c^2)]\gamma_j$, the random vector $\vec{R}$ as a sum of independent random variables $\vec{r}_j$ has the distribution

$$w(\vec{R}) = \frac{1}{2\pi \sum_j \sigma_j^2} \exp\left(-\frac{R_x^2 + R_y^2}{2\sum_j \sigma_j^2}\right), \qquad (40)$$

$$\sigma^2 = \sum_j \sigma_j^2 = 2q\sum_j \Delta t \frac{1}{\beta^2 + \omega_c^2}\gamma_j. \qquad (41)$$

In the limit of $\Delta t \to 0$ we arrive at the integral

$$\sigma^2 = 2q\frac{1}{\beta^2 + \omega_c^2}\int_0^t \gamma(s)ds \qquad (42)$$

with $\int_0^t \gamma(s)ds = t + \Theta$, where

$$\Theta = \Theta(t) = \frac{1}{2\beta}(1 - e^{-2\beta t}) - 2\frac{1}{\beta^2 + \omega_c^2}[\beta + (\omega_c \sin \omega_c t$$

$$- \beta \cos \omega_c t)e^{-\beta t}]. \qquad (43)$$

That gives rise to an ultimate form of the transition probability density of the spatial displacement process

$$P(\vec{r},t|\vec{r}_0,t_0=0,\vec{u}_0) = \frac{1}{4\pi \dfrac{k_B T}{m}\dfrac{\beta}{\beta^2 + \omega_c^2}(t+\Theta)}$$

$$\times \exp\left(-\frac{|\vec{r} - \vec{r}_0 - \Omega\vec{u}_0|^2}{4\dfrac{k_B T}{m}\dfrac{\beta}{\beta^2 + \omega_c^2}(t+\Theta)}\right) \qquad (44)$$

with $\Omega = \Omega(t)$ defined in Eqs. (25) and (27). Notice that an asymptotic diffusion coefficient $D_B = D\beta^2/(\beta^2 + \omega_c^2)$ of Sec. II [cf. Eq. (20)] appears here as a spatial dispersion-attenuation signature (when ω_c grows up at fixed β).

The spatial displacement process governed by the above transition probability density surely is *not* Markovian. That can be checked by inspection: the Chapman-Kolmogorov identity is not valid, like in the standard free Brownian motion example where the Ornstein-Uhlenbeck process induced (sole) spatial dynamics is non-Markovian as well.

IV. PHASE-SPACE PROCESS

A. Free Brownian motion, Kramers equation, and local conservation laws

We take advantage of the cylindrical symmetry of our problem, and consider separately the (free) Brownian dynamics in the direction parallel to the magnetic field vector, e.g., along the Z axis. That amounts to invoking a familiar Ornstein-Uhlenbeck process (in velocity/momentum) in its extended phase-space form. In the absence of external forces, the kinetic (Kramers-Fokker-Planck equation) reads

$$\partial_t W + u\nabla_z W = \beta \nabla_u(Wu) + q\triangle_u W, \qquad (45)$$

where $q = D\beta^2$. Here β is the friction coefficient, D will be identified later with the spatial diffusion constant, and (as before) we set $D = k_B T/m\beta$ in conformity with the Einstein fluctuation-dissipation identity.

The joint probability distribution (in fact, density) $W(z,u,t)$ for a freely moving Brownian particle that at $t = 0$ initiates its motion at x_0 with an arbitrary inital velocity u_0 can be given in the form of the maximally symmetric displacement probability law

$$W(z,u,t) = W(R,S) = [4\pi^2(FG - H^2)]^{-1/2}$$

$$\times \exp\left\{-\frac{GR^2 - HRS + FS^2}{2(FG - H^2)}\right\}, \qquad (46)$$

where $R = z - u_0(1 - e^{-\beta t})\beta^{-1}$, $S = u - u_0 e^{-\beta t}$ while $F = (D/\beta)(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t})$, $G = D\beta(1 - e^{-2\beta t})$, and $H = D(1 - e^{-\beta t})^2$.

For future reference, let us notice that marginal probablity densities, in the Smoluchowski regime [take for granted that time scales β^{-1} and space scales $(D\beta^{-1})^{1/2}$ are irrelevant [12]] display familiar forms of the Maxwell-Boltzmann probability density $w(u,t) = (m/2\pi kT)^{1/2}\exp(-mu^2/2k_B T)$ and the diffusion kernel $w(z,t) = (4\pi Dt)^{-1/2}\exp(-z^2/4Dt)$, respectively.

A direct evaluation of the first and second local moment of the phase-space probability density gives

$$\langle u\rangle = \int du\, u W(z,u,t) = w(R)[(H/F)R + u_0 e^{-\beta t}], \qquad (47)$$

$$\langle u^2\rangle = \int du\, u^2 W(z,u,t) = \left(\frac{FG - H^2}{F} + \frac{H^2}{F^2}R^2\right)$$

$$\times (2\pi F)^{-1/2}\exp\left(-\frac{R^2}{2F}\right). \qquad (48)$$

We notice that after passing to the diffusion (Smoluchowski) regime [7], one readily recovers the local (configuration space conditioned) moment $\langle u\rangle_z = (1/w)\langle u\rangle$ to be in the form

$$\langle u\rangle_z = \frac{z}{2t} = -D\frac{\nabla w(z,t)}{w(z,t)}, \qquad (49)$$

while for the second local moment $\langle u^2 \rangle_z = (1/w)\langle u^2 \rangle$ we would arrive at

$$\langle u^2 \rangle_z = (D\beta - D/2t) + \langle u \rangle_z^2. \tag{50}$$

By inspection one verifies that the transport (Kramers) equation for $W(z,u,t)$ implies local conservation laws

$$\partial_t w + \nabla(\langle u \rangle_z w) = 0 \tag{51}$$

and

$$\partial_t(\langle u \rangle_z w) + \nabla_z(\langle u^2 \rangle_z w) = -\beta \langle u \rangle_z w. \tag{52}$$

At this point (we strictly follow the moment equations strategy of the traditional kinetic theory of gases and liquids, compare, e.g., [11]) let us introduce the notion of the pressure function P_{kin}

$$P_{kin}(z,t) = (\langle u^2 \rangle_z - \langle u \rangle_z^2) w(z,t), \tag{53}$$

in terms of which we can analyze the local momentum conservation law

$$(\partial_t + \langle u \rangle_z \nabla)\langle u \rangle_z = -\beta \langle u \rangle_z - \frac{\nabla P_{kin}}{w}. \tag{54}$$

One should realize that in the Smoluchowski regime the friction term is canceled away by a counterterm coming from $(1/w)\nabla P_{kin}$ so that

$$(\partial_t + \langle u \rangle_z \nabla)\langle u \rangle_z = \frac{D}{2t}\frac{\nabla w}{w} = -\frac{\nabla P}{w}, \tag{55}$$

where $P = D^2 w \triangle \ln w$, called osmotic pressure in Ref. [13], is the net remnant of the kinetic pressure contribution.

Further exploiting the kinetic lore, we can tell few words about the *temperature of Brownian particles* as opposed to the (equilibrium) temperature of the thermal bath. Namely, in view of (we refer to the Smoluchowski regime with $t \gg \beta^{-1}$) $P_{kin} \sim (D\beta - D/2t)w$ where $D = k_B T/m\beta$, we can formally set

$$k_B T_{kin} = \frac{P_{kin}}{w} \sim \left(k_B T - \frac{D}{2t}\right) < k_B T. \tag{56}$$

That quantifies the degree of thermal agitation (temperature) of Brownian particles to be *less* than the thermostat temperature. Heat is continually pumped from the thermostat to the Brownian ''gas,'' until asymptotically both temperatures equalize. This may be called a ''thermalization'' of Brownian particles. In the process of that ''thermalization'' the Brownian ''gas'' temperature monotonically grows up until the mean kinetic energy of particles and that of mean-flows asymptotically approach the familiar kinetic relationship: $\int (w/2)(\langle u^2 \rangle_z - \langle u \rangle_z^2)dx = k_B T$, cf. Refs. [6,7] for more extended discussion of that medium → particles heat transfer issue and its possible relevance while associating habitual thermal equilibrium conditions with essentially nonequilibrium phenomena.

Remark 1. Once local conservation laws were introduced, it seems instructive to comment on the essentially hydrodynamical features (compressible fluid/gas case) of the problem. Specifically, the ''pressure'' term ∇Q is quite annoying from the traditional kinetic theory perspective. That is apart from the fact that our local conservation laws have a conspicuous Euler form appropriate for the standard hydrodynamics of gases and liquids. One should become alert that in the present (Brownian) context they convey an entirely different message. For example, in case of normal liquids the pressure is exerted upon any control volume (droplet) by the surrounding fluid. We may interpret that as a compression of a droplet. In the case of Brownian motion, we deal with a definite decompression: particles are driven away from areas of higher concentration (probability of occurence). Hence, typically the Brownian ''pressure'' is exerted by the droplet upon its surrounding.

Remark 2. The derivation of a hierarchy of local conservation laws (moment equations) for the Kramers equation can be patterned after the standard procedure for the Boltzmann equation. Those laws do not form a closed system and additional specifications (like the familiar thermodynamic equation of state) are needed to that end. In case of the isothermal Brownian motion, when considered in the large friction regime (e.g., Smoluchowski diffusion approximation), it suffices to supplement the Fokker-Planck equation by one more conservation law *only* to arrive at a closed system [6] and compare with the discussion of Ref. [11].

B. Planar process

Now we shall consider Brownian dynamics in the direction perpendicular to the magnetic field $\vec{B}$, hence (while in terms of configuration-space variables) we address an issue of the planar dynamics. We are interested in the complete phase-space process, hence we need to specify the transition probability density $P(\vec{r},\vec{u},t|\vec{r}_0,\vec{u}_0,t_0=0)$ of the Markov process conditioned by the initial data $\vec{u} = \vec{u}_0$ and $\vec{r} = \vec{r}_0$ at time $t_0 = 0$. That is equivalent to deducing the joint probability distribution $W(\vec{S},\vec{R})$ of random vectors $\vec{S}$ and $\vec{R}$, previously defined to appear in the form $\vec{S} = \vec{u}(t) - e^{-\Lambda t}\vec{u}_0$ and $\vec{R} = \vec{r} - \vec{r}_0 - \Omega\vec{u}_0$, respectively, cf. Eqs. (15) and (44).

Let us stress that presently, all vectors are regarded as two-dimensional versions (the third component being simply disregarded) of the original random variables we have discussed so far in Sec. II and III. Vectors $\vec{S}$ and $\vec{R}$ both share a Gaussian distribution with mean zero. Consequently, the joint distribution $W(\vec{S},\vec{R})$ is determined by the matrix of variances and covariances: $C = (c_{ij}) = (\langle x_i x_j \rangle)$, where we abbreviate four phase-space variables in a single notion of $x = (S_1, S_2, R_1, R_2)$ and label components of x by $i,j = 1,2,3,4$. In terms of $\vec{R}$ and $\vec{S}$ the covariance matrix C reads

$$C = \begin{vmatrix} \langle S_1 S_1 \rangle & \langle S_1 S_2 \rangle & \langle S_1 R_1 \rangle & \langle S_1 R_2 \rangle \\ \langle S_2 S_1 \rangle & \langle S_2 S_2 \rangle & \langle S_2 R_1 \rangle & \langle S_2 R_2 \rangle \\ \langle R_1 S_1 \rangle & \langle R_1 S_2 \rangle & \langle R_1 R_1 \rangle & \langle R_1 R_2 \rangle \\ \langle R_2 S_1 \rangle & \langle R_2 S_2 \rangle & \langle R_2 R_1 \rangle & \langle R_2 R_2 \rangle \end{vmatrix}. \tag{57}$$

The joint probability distribution of $\vec{S}$ and $\vec{R}$ is given by

$$W(\vec{S},\vec{R})=W(\vec{x})=\frac{1}{4\pi^2}\left(\frac{1}{\det C}\right)^{1/2}\exp\left(-\frac{1}{2}\sum_{i,j}c_{ij}^{-1}x_ix_j\right),$$

(58)

where c_{ij}^{-1} denotes the component of the inverse matrix C^{-1}.

The probability distributions of $\vec{S}$ and $\vec{R}$, which were established in the previous sections, determine a number of expectation values:

$$g\equiv\langle S_1S_1\rangle=\langle S_2S_2\rangle=\frac{k_BT}{m}(1-e^{-2\beta t}),$$

(59)

cf. Eq. (14), while $\langle S_1S_2\rangle=\langle S_2S_1\rangle=0$. Furthermore,

$$f\equiv\langle R_1R_1\rangle=\langle R_2R_2\rangle=2\frac{k_BT}{m}\frac{\beta}{\beta^2+\omega_c^2}(t+\Theta)=2D_B(t+\Theta),$$

(60)

cf. Eqs. (20), (42), (43). In addition we have $\langle R_1R_2\rangle=\langle R_2R_1\rangle=0$.

As a consequence, we are left with only four nonvanishing components of the covariance matrix C: $c_{13}=c_{31}=\langle S_1R_1\rangle$, $c_{14}=c_{41}=\langle S_1R_2\rangle$, $c_{23}=c_{32}=\langle S_2R_1\rangle$, $c_{24}=c_{42}=\langle S_2R_2\rangle$, which need a closer examination.

We can obtain those covariances by exploiting a dependence of the random quantities $\vec{S}$ and $\vec{R}$ on the white-noise term $\vec{A}(s)$ whose statistical properties are known. There follows

$$S_1=\int_0^t ds\, e^{-\beta(t-s)}[\cos\omega_c(t-s)A_1(s)$$

$$+\sin\omega_c(t-s)A_2(s)],$$

(61)

$$S_2=\int_0^t ds\, e^{-\beta(t-s)}[-\sin\omega_c(t-s)A_1(s)$$

$$+\cos\omega_c(t-s)A_2(s)],$$

$$R_1=\int_0^t ds\,\frac{1}{\beta^2+\omega_c^2}[\beta\{1-e^{-\beta(t-s)}\cos\omega_c(t-s)\}$$

$$+\omega_c e^{-\beta(t-s)}\sin\omega_c(t-s)]A_1(s)$$

$$+\int_0^t ds\,\frac{1}{\beta^2+\omega_c^2}[-\beta e^{-\beta(t-s)}\sin\omega_c(t-s)$$

$$+\omega_c\{1-e^{-\beta(t-s)}\cos\omega_c(t-s)\}]A_2(s),$$

$$R_2=\int_0^t ds\,\frac{1}{\beta^2+\omega_c^2}[-\omega_c\{1-e^{-\beta(t-s)}\cos\omega_c(t-s)\}$$

$$+\beta e^{-\beta(t-s)}\sin\omega_c(t-s)]A_1(s)$$

$$+\int_0^t ds\,\frac{1}{\beta^2+\omega_c^2}[\omega_c e^{-\beta(t-s)}\sin\omega_c(t-s)$$

$$+\beta\{1-e^{-\beta(t-s)}\cos\omega_c(t-s)\}]A_2(s).$$

Multiplying together suitable components of vectors $\vec{S}$ and $\vec{R}$ and taking averages of those products in conformity with the rules $\langle A_i(s)\rangle=0$ and $\langle A_i(s)A_j(s')\rangle=2q\delta_{ij}\delta(s-s')$, where $q=(k_BT/m)\beta$, $i,j=1,2,3$, we arrive at

$$h\equiv\langle R_1S_1\rangle=\langle R_2S_2\rangle=2q\frac{1}{\beta^2+\omega_c^2}\int_0^t ds\, e^{-\beta(t-s)}$$

$$\times[\beta\cos\omega_c(t-s)+\omega_c\sin\omega_c(t-s)-\beta e^{-\beta(t-s)}]$$

$$=q\frac{1}{\beta^2+\omega_c^2}(1-2e^{-\beta t}\cos\omega_c t+e^{-2\beta t})$$

(62)

and

$$k\equiv\langle R_1S_2\rangle=-\langle R_2S_1\rangle=2q\frac{1}{\beta^2+\omega_c^2}\int_0^t ds\, e^{-\beta(t-s)}$$

$$\times[-\beta\sin\omega_c(t-s)+\omega_c\cos\omega_c(t-s)-\omega_c e^{-\beta(t-s)}]$$

$$=q\frac{1}{\beta^2+\omega_c^2}\left[2e^{-\beta t}\sin\omega_c t-\frac{\omega_c}{\beta}(1-e^{-2\beta t})\right].$$

(63)

The covariance matrix $C=(c_{ij})$ has thus the form

$$C=\begin{pmatrix} g & 0 & h & -k \\ 0 & g & k & h \\ h & k & f & 0 \\ -k & h & 0 & f \end{pmatrix}$$

(64)

while its inverse C^{-1} reads as follows:

$$C^{-1}=\frac{1}{\det C}(fg-h^2-k^2)\begin{pmatrix} f & 0 & -h & k \\ 0 & f & -k & -h \\ -h & -k & g & 0 \\ k & -h & 0 & g \end{pmatrix},$$

(65)

where $\det C=(fg-h^2-k^2)^2$.

The joint probability distribution of $\vec{S}$ and $\vec{R}$ can be ultimately written in the form

$$W(\vec{S},\vec{R}) = \frac{1}{4\pi^2(fg-h^2-k^2)}$$

$$\times \exp\left(-\frac{f|\vec{S}|^2+g|\vec{R}|^2-2h\vec{S}\cdot\vec{R}+2k(\vec{S}\times\vec{R})_{i=3}}{2(fg-h^2-k^2)}\right).$$

$$(66)$$

In the above, all vector entries are two dimensional. The specific $i=3$ vector product coordinate in the exponent is simply an abbreviation for the (ordinary R^3-vector product) procedure that involves merely the first two components of three-dimensional vectors (the third is then arbitrary and irrelevant), hence effectively involves our two-dimensional $\vec{R}$ and $\vec{S}$.

C. Kramers equation and local conservation laws for the planar motion

For the purpose of evaluating local velocity averages of the Kramers equation, we need to extract the marginal configuration space distribution. Let us notice that

$$\int W(\vec{S},\vec{R})d\vec{S} = w(\vec{R}) = P(\vec{r},t|\vec{r}_0,t_0=0,\vec{u}_0), \quad (67)$$

where the last transition probability density entry coincides with that of Eq. (44).

Let us introduce an auxiliary (weighted) distribution:

$$\widetilde{W}(\vec{S}|\vec{R}) = \frac{W(\vec{S},\vec{R})}{\int W(\vec{S},\vec{R})d\vec{S}} = \frac{1}{2\pi\frac{1}{f}(fg-h^2-k^2)}$$

$$\times \exp\left(-\frac{|\vec{S}-\vec{m}|^2}{2\frac{1}{f}(fg-h^2-k^2)}\right), \quad (68)$$

where

$$\vec{m} = \frac{1}{f}(hR_1-kR_2, hR_2+kR_1) \quad (69)$$

and we recall that $\vec{S} = \vec{u}(t) - e^{-\beta t}U(t)\vec{u}_0$ and $\vec{R} = \vec{r} - \vec{r}_0 - \Omega\vec{u}_0$.

The local expectation values (compare, e.g., calculations of the previous subsection) read: $\langle u_i\rangle_{\vec{R}} = \int u_i\widetilde{W}d\vec{u}$ and $\langle u_i^2\rangle_{\vec{R}} = \int u_i^2\widetilde{W}d\vec{u}$ where $i=1,2$. By evaluating those averages we get

$$\langle\vec{u}\rangle_{\vec{R}} = (\langle u_1\rangle_{\vec{R}}, \langle u_2\rangle_{\vec{R}}) = e^{-\beta t}U(t)\vec{u}_0 + \vec{m} \quad (70)$$

and

$$\langle u_1^2\rangle_{\vec{R}} - \langle u_1\rangle_{\vec{R}}^2 = \langle u_2^2\rangle_{\vec{R}} - \langle u_2\rangle_{\vec{R}}^2 = \frac{1}{f}(fg-h^2-k^2). \quad (71)$$

The Fokker-Planck-Kramers equation, appropriate for the planar dynamics in its phase space version, reads

$$\frac{\partial W}{\partial t} + \vec{u}\vec{\nabla}_{\vec{r}}W = \beta\vec{\nabla}_{\vec{u}}(W\vec{u}) - \omega_c[\vec{\nabla}_{\vec{u}}\times W\vec{u}]_{i=3} + q\vec{\nabla}_{\vec{u}}^2W,$$

$$(72)$$

where again the troublesome (in the planar case all vectors are two-dimensional) vector product third component stands for

$$[\vec{\nabla}_{\vec{u}}\times W\vec{u}]_{i=3} = \frac{\partial}{\partial u_1}(Wu_2) - \frac{\partial}{\partial u_2}(Wu_1). \quad (73)$$

The first two moment equations are easily derivable. Namely, the continuity (zeroth moment) and the momentum conservation (first moment) equations come out in the form

$$\partial_t w + \vec{\nabla}\cdot[\langle\vec{u}\rangle_{\vec{R}}w] = 0 \quad (74)$$

and

$$\partial_t[\langle u_1\rangle_{\vec{R}}w] + \frac{\partial}{\partial r_1}[\langle u_1^2\rangle_{\vec{R}}w] + \frac{\partial}{\partial r_2}[\langle u_1\rangle_{\vec{R}}\langle u_2\rangle_{\vec{R}}w]$$

$$= -\beta\langle u_1\rangle_{\vec{R}}w + \omega_c\langle u_2\rangle_{\vec{R}}w \quad (75)$$

$$\partial_t[\langle u_2\rangle_{\vec{R}}w] + \frac{\partial}{\partial r_2}[\langle u_2^2\rangle_{\vec{R}}w] + \frac{\partial}{\partial r_1}[\langle u_1\rangle_{\vec{R}}\langle u_2\rangle_{\vec{R}}w]$$

$$= -\beta\langle u_2\rangle_{\vec{R}}w - \omega_c\langle u_1\rangle_{\vec{R}}w.$$

That implies

$$\left[\partial_t + \langle u_1\rangle_{\vec{R}}\frac{\partial}{\partial r_1} + \langle u_2\rangle_{\vec{R}}\frac{\partial}{\partial r_2}\right]\langle u_1\rangle_{\vec{R}}$$

$$= -\beta\langle u_1\rangle_{\vec{R}} + \omega_c\langle u_2\rangle_{\vec{R}} - \frac{1}{w}\frac{\partial}{\partial r_1}[(\langle u_1^2\rangle_{\vec{R}} - \langle u_1\rangle_{\vec{R}}^2)w],$$

$$(76)$$

$$\left[\partial_t + \langle u_1\rangle_{\vec{R}}\frac{\partial}{\partial r_1} + \langle u_2\rangle_{\vec{R}}\frac{\partial}{\partial r_2}\right]\langle u_2\rangle_{\vec{R}}$$

$$= -\beta\langle u_2\rangle_{\vec{R}} - \omega_c\langle u_1\rangle_{\vec{R}} - \frac{1}{w}\frac{\partial}{\partial r_2}[(\langle u_2^2\rangle_{\vec{R}} - \langle u_2\rangle_{\vec{R}}^2)w],$$

which finally sums up to a local momentum conservation law (here, the standard R^3 vector product on the right-hand side contributes its first and second components only)

$$[\partial_t + \langle\vec{u}\rangle_{\vec{R}}\vec{\nabla}]\langle\vec{u}\rangle_{\vec{R}} = -\Lambda\langle\vec{u}\rangle_{\vec{R}} - \frac{1}{w}\vec{\nabla}\cdot\vec{P}_{kin} = -\beta\langle\vec{u}\rangle_{\vec{R}}$$

$$+ \frac{q_e}{mc}\langle\vec{u}\rangle_{\vec{R}}\times\vec{B} - \frac{1}{w}\vec{\nabla}\cdot\vec{P}_{kin}, \quad (77)$$

where $\vec{P}_{kin}$ has tensor components P_{ij}^{kin}, and $\vec{\nabla}\cdot\vec{P}_{kin}$ stands for a vector whose ith component is equal $\sum_j\partial P_{ij}^{kin}/\partial r_j$ and

$i, j = 1, 2$. Here, obviously $P_{ij}^{kin} = \langle (u_i - \langle u_i \rangle_{\vec{R}})(u_j - \langle u_j \rangle_{\vec{R}}) \rangle_{\vec{R}} w$ and only diagonal entries do not vanish. Clearly $P_{ii}^{kin} = (\langle u_i^2 \rangle_{\vec{R}} - \langle u_i \rangle_{\vec{R}}^2) w$.

Because of

$$\sigma^2 = \langle u_1^2 \rangle_{\vec{R}} - \langle u_1 \rangle_{\vec{R}}^2 = \langle u_2^2 \rangle_{\vec{R}} - \langle u_2 \rangle_{\vec{R}}^2$$

$$= \frac{1}{f}(fg - h^2 - k^2)$$

$$= g - \frac{h^2 + k^2}{f}, \tag{78}$$

we can introduce again $\sigma^2 = P_{kin}/w = k_B T_{kin}$ and pass to an asymptotic regime $t \gg t_c = 1/\beta$. Then, we obtain Eq. (56) to be valid in the present case as well thus quantifying an overall (magnetic field independent) heating process involved.

In that asymptotic regime we have $\sigma^2 = D\beta - D/2t$ and by employing an asymptotic form of $w(\vec{R})$, Eq. (44) we recover

$$\vec{\nabla} \cdot \vec{P}_{kin} = \left(D\beta - \frac{D}{2t} \right) \frac{\vec{\nabla} w}{w} \tag{79}$$

together with

$$\Lambda \langle \vec{u} \rangle_{\vec{R}} = -D\beta \frac{\vec{\nabla} w}{w}. \tag{80}$$

So, asymptotically $(t \gg \beta^{-1})$ the momentum conservation law takes the form (to be compared with considerations of Sec. IV A

$$[\partial_t + \langle \vec{u} \rangle_{\vec{R}} \vec{\nabla}] \langle \vec{u} \rangle_{\vec{R}} = \frac{D}{2t} \frac{\vec{\nabla} w}{w}. \tag{81}$$

However an asymptotic regime does not yet imply that the right-hand side of Eq. (82) represents an acceptable ''osmotic pressure'' gradient contribution. We additionally need a large friction regime to deal with a consistent picture of a Markov diffusion process in the Smoluchowski form. In-

deed, to reproduce a universal (Ref. [6]) pressure-type functional dependence on P we must employ a suitable form of the diffusion coefficient. The usage of D alone to define an ''osmotic pressure'' implies an apparent failure. On the other hand, the usage of $D_B = D\beta^2/(\beta^2 + \omega_c^2)$ as suggested by Eq. (44) leads to

$$P = D_B^2 \Delta \ln w \Rightarrow -\frac{\vec{\nabla} P}{w} = \frac{D_B}{2t} \frac{\vec{\nabla} w}{w}. \tag{82}$$

Then, however, the momentum conservation law displays a supplementary scaling of the osmotic pressure contribution, which may trivialize (die out) only for large values of β (an ultimate Smoluchowski regime). Namely, we have

$$[\partial_t + \langle \vec{u} \rangle_{\vec{R}} \vec{\nabla}] \langle \vec{u} \rangle_{\vec{R}} = -\left(\frac{\beta^2}{\beta^2 + \omega_c^2} \right)^{-1} \frac{\vec{\nabla} P}{w}. \tag{83}$$

Such process is yet non-Markovian and its approximation by the Smoluchowski process becomes reliable when β is large while ω_c is kept moderately small.

Basically, the large friction regime cancels all rotational features (arising due to the Lorentz force) on very short time scales. If we are satisfied with the non-Markovian regime of moderate friction but arbitrarily varying ω_c (i.e., B) then, in conformity with Eq. (77), mean flows would display signatures of rotation that are bound to die out asymptotically. This effect can be analyzed in R^3 by observing that a three-dimensional extension of the vector $\vec{m}$ of Eqs. (69) and (70) asymptotically reads $\vec{m} = (1/2t\beta)\Lambda(\vec{r} - \vec{r}_0)$ where Λ comes from Eq. (3) and $(\vec{r} - \vec{r}_0) \in R^3$. In view of that, we have [cf. Eq. (70)] $\mathrm{curl} \langle \vec{u} \rangle_{\vec{R}} \sim \mathrm{curl} \vec{m} \sim (0, 0, -\omega_c/t\beta)$ and the circulation asymptotically vanishes. The effect can be slowed down by a suitable adjustment of ω_c against β.

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